Riccati parameter modes from Newtonian free damping motion by supersymmetry

Haret C. Rosu and Marco A. Reyes
Instituto de Física de la Universidad de Guanajuato, Apartado Postal E-143, León, Guanajuato, Mexico
(Received 31 July 1997)

We determine the class of damped modes $\tilde{y}$ that are related to the common free damping modes y by supersymmetry. They are obtained by employing the factorization of Newton's differential equation of motion for the free damped oscillator by means of the general solution of the corresponding Riccati equation together with Witten's method of constructing the supersymmetric partner operator. This procedure leads to one-parameter families of (transient) modes for each of the three types of free damping, corresponding to a particular type of antirestoring acceleration (adding up to the usual Hooke restoring acceleration) of the form $a(t)=2\gamma^2/(\gamma t+1)^2\tilde{y}$, where γ is the family parameter that has been chosen as the inverse of the Riccati integration constant. In supersymmetric terms, they represent all those one-Riccati-parameter damping modes having the same Newtonian free damping partner mode. [S1063-651X(98)00604-7]

PACS number(s): 03.20.+i

The damped oscillator (DO) is a cornerstone of physics and a primary textbook example in classical mechanics. Schemes of analogies allow its extension to many areas of physics where the same basic concepts occur with merely a change in the meaning of the symbols. Apparently, there might hardly be anything new to say about such an obvious case. However, in the following we would like to exhibit a different and nice feature of damping resulting from the mathematical procedure of factorization of its differential equation. In the past, the factorization of the DO differential equation (Newton's law) has been tackled by a few authors [1], but not in the framework that will be presented herein. Namely, recalling that such factorizations are common tools in Witten's supersymmetric quantum mechanics [2] and imply particular solutions of Riccati equations known as superpotentials, we would like to explore here the factoring of the DO equation by means of the general solution of the Riccati equation, a procedure that has been used in physics by Mielnik [3] for the quantum harmonic oscillator. In other words, our goal here is to exploit the nonuniqueness of the factorization of second-order differential operators, on the example of the classical damped oscillator. By doing this one may hope to gain insight into the free damping motion. We write the ordinary DO Newton's law in the form

$$Ny\equiv\left(\frac{d^2}{dt^2}+2\beta\frac{d}{dt}+\beta^2\right)y=(\beta^2-\omega_0^2)y=\alpha^2y, \quad (1)$$

i.e., we already added a β^2y term on both sides in order to perform the factoring. The coefficient 2β is the friction constant per unit mass and ω_0 is the natural frequency of the oscillator. The factorization

$$\left(\frac{d}{dt}+\beta\right)\left(\frac{d}{dt}+\beta\right)y=\alpha^2y \quad (2)$$

follows, and previous authors [1] discussed the classical cases of underdamping ($\alpha^2<0$), critical damping ($\alpha^2=0$), and overdamping ($\alpha^2>0$) in terms of the first-order differential equation

$$Ly\equiv\left(\frac{d}{dt}+\beta\right)y_{\pm}=\pm\alpha y_{\pm}. \quad (3)$$

It follows that $y_{\pm}=e^{-\beta t\pm\alpha t}$ and one can build through their superposition the general solution as $y=e^{-\beta t}(Ae^{\alpha t}+Be^{-\alpha t})$. Thus, for free underdamping, the general solution can be written as $y_u=\tilde{A}e^{-\beta t}\cos(\sqrt{-\alpha^2}t+\phi)$, where $\tilde{A}=2\sqrt{|AB|}$ and $\phi=\arccos[(A+B)/\tilde{A}]$, whereas the overdamped general solution is $\tilde{A}e^{-\beta t}\cosh(\alpha t+\phi)$, where $\tilde{A}=2\sqrt{|AB|}$ and $\phi=\arccos[(A+B)/\tilde{A}]$. The critical case is special but well known [1], having the general solution of the type $y_c=e^{-\beta t}(A+Bt)$.

Since, as we mentioned, the factorization given by Eq. (2) may not be the only one possible, let us now write the more general factorization

$$N_gy\equiv\left(\frac{d}{dt}+f(t)\right)\left(\frac{d}{dt}+g(t)\right)y=\alpha^2y, \quad (4)$$

where $f(t)$ and $g(t)$ are two functions of time. The condition that N_g be identical to N leads to $f(t)+g(t)=2\beta$ and $g'+fg=\beta$, which can be combined in the Riccati equation

$$-f'-f^2+2\beta f=\beta^2. \quad (5)$$

By inspection, one can easily see that a first solution to this equation is $f(t)=\beta$ [$g(t)=\beta$], which is the common case discussed by previous authors [1]. Changing the dependent variable to $h(t)=f(t)-\beta$, we get a simpler form of the Riccati equation, i.e., $h'(t)+h^2=0$, with the particular solution $h(t)=0$. However, the general solution is $h(t)=1/(t+T)=\gamma/(\gamma t+1)$, as one can easily check. The constant of integration $T=1/\gamma$ occurs as a new time scale in the problem; see below. Therefore, there is the more general factorization of the DO equation than Eq. (2):

$$A^+A^-y\equiv\left(\frac{d}{dt}+\beta+\frac{\gamma}{\gamma t+1}\right)\left(\frac{d}{dt}+\beta-\frac{\gamma}{\gamma t+1}\right)y=\alpha^2y. \quad (6)$$

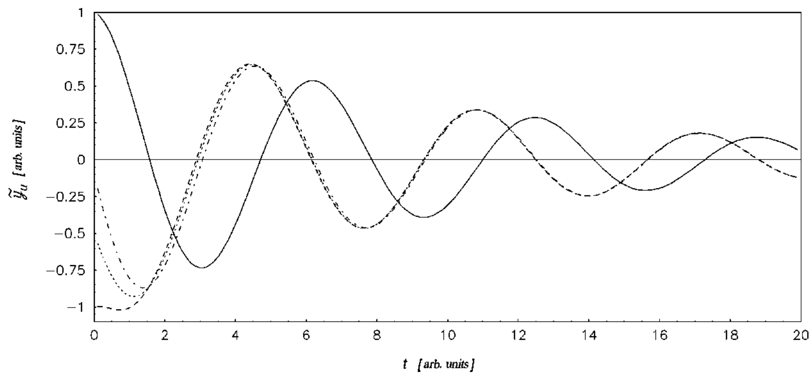

FIG. 1. Initial free underdamped mode of the type $y_u=e^{-t/10}\cos t$ (bold curve) and members of its γ family of supersymmetric damping modes $\tilde{y}_u=-e^{-t/10}[\sin t+\gamma/(\gamma+1)\cos t]$ for the following values of parameter γ: 1, dashed curve; $\frac{1}{2}$, bold dashed curve; $\frac{1}{4}$, solid curve.

A few remarks are in order. While the linear operator $L=d/dt+\beta$ has $y_\pm$ as eigenfunctions with eigenvalues $\pm\alpha$, the quadratic operator N has $y_\pm$ as degenerate eigenfunctions, with the same eigenvalue α^2. On the other hand, the new linear operators A^+ and A^- do not have $y_\pm$ as eigenfunctions since $A^+y_\pm=[\pm\alpha+\gamma/(\gamma t+1)]y_\pm$ and $A^-y_\pm=[\pm\alpha-\gamma/(\gamma t+1)]y_\pm$, although the quadratic operator $N_g=A^+A^-$ still has $y_\pm$ as degenerate eigenfunctions at eigenvalue α^2. We now construct, according to the ideas of supersymmetric quantum mechanics [2], the supersymmetric partner of N_g,

$$\tilde{N}_g=A^-A^+=\frac{d^2}{dt^2}+2\beta\frac{d}{dt}+\beta^2-\frac{2\gamma^2}{(\gamma t+1)^2}. \qquad (7)$$

This second-order damping operator contains the additional last term with respect to its initial partner, which, roughly speaking, is the Darboux transform term [4] of the quadratic operator. The important property of this operator is the following. If y_0 is an eigenfunction of N_g, then A^-y_0 is an eigenfunction of $\tilde{N}_g$ since $\tilde{N}_gA^-y_0=A^-A^+A^-y_0=A^-N_gy_0$ and $N_gy_0=\alpha^2y_0$, implying $\tilde{N}_g(A^-y_0)=A^-N_gy_0=\alpha^2(A^-y_0)$. The conclusion is that $\tilde{N}_g$ has the same type of "spectrum" as N_g and therefore as N. The eigenfunctions $\tilde{y}_\pm$ can be constructed if one knows the eigenfunctions $y_\pm$ as

$$\tilde{y}_\pm=A^-y_\pm=\left(\frac{d}{dt}+\beta-\frac{\gamma}{\gamma t+1}\right)y_\pm \qquad (8)$$

and thus

$$\tilde{y}_\pm=\left(\pm\alpha-\frac{\gamma}{\gamma t+1}\right)e^{-\beta t\pm\alpha t}. \qquad (9)$$

These modes make up a one-parameter family of damping eigenfunctions that we interpret as follows. We write down the usual form of the Newton law corresponding to the Newton operator $\tilde{N}_g$,

$$\left(\frac{d^2}{dt^2}+2\beta\frac{d}{dt}+\omega_0^2-\frac{2\gamma^2}{(\gamma t+1)^2}\right)\tilde{y}=0. \qquad (10)$$

Examination of this law shows that the term $2\gamma^2/(\gamma t+1)^2\tilde{y}$ can be interpreted as a time-dependent antirestoring acceleration (because of the minus sign in front of it) producing in the transient period $t\leqslant 1/\beta$ the damping modes given by $\tilde{y}$ above.

We present now separately the $\tilde{y}$ families of modes calculated as superpositions of the modes $\tilde{y}_\pm$ for the three types of free damping.

(i) For underdamping $\beta^2<\omega_0^2$, let $\alpha=i\omega_1$, where $\omega_1=\sqrt{\omega_0^2-\beta^2}$. The original eigenfunction is $y_u=\tilde{A}_u\cos(\omega_1 t+\phi)e^{-\beta t}$, while the supersymmetric family is $\tilde{y}_u=-\tilde{A}_u(\omega_1\sin(\omega_1 t+\phi)+[\gamma/(\gamma t+1)]\cos(\omega_1 t+\phi))e^{-\beta t}$.

(ii) In the case of critical damping $\beta^2=\omega_0^2$, the general free solution is $y_c=Ae^{-\beta t}+Bte^{-\beta t}$, whereas the tilde solu-

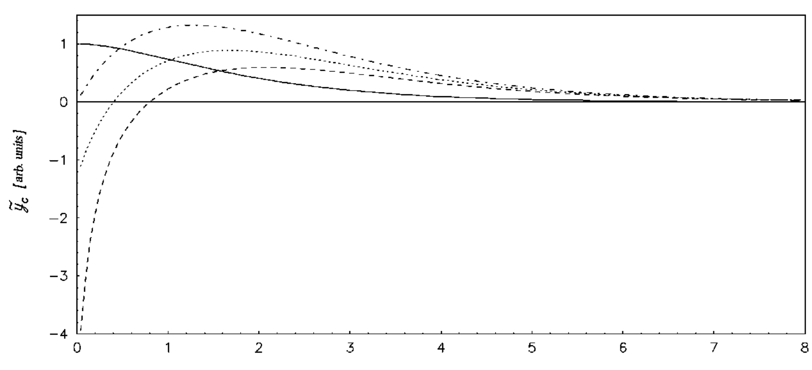

FIG. 2. Initial free critical damping mode $y_c=e^{-t}(1+t)$ (bold curve) and members of the corresponding γ family $\tilde{y}_c=e^{-t}[-\gamma/(\gamma t+1)+(\gamma t+1)^2/\gamma^2]$ for the γ parameter taking the following values: 1, dashed curve; 2, bold dashed curve; 4, solid curve.

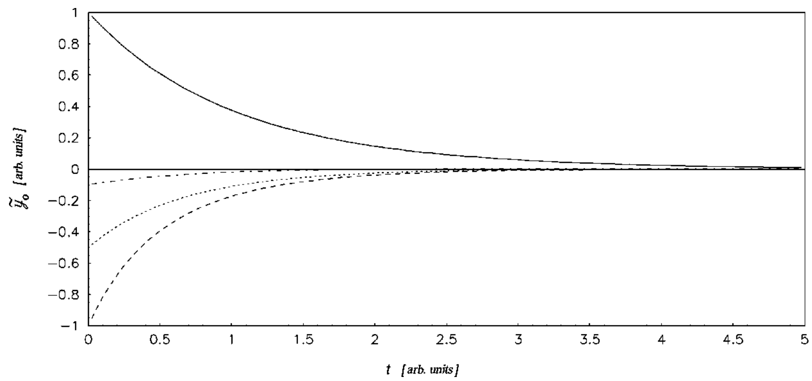

FIG. 3. Initial free overdamped mode of the type $y_0 = e^{-t}\cosh(t/5)$ (bold curve) and members of its supersymmetric γ family $\tilde{y}_0 = e^{-t}(\frac{1}{5}\sinh(t/5) - [\gamma/(\gamma t+1)]\cosh(t/5))$ for the following values of the parameter γ: 1, dashed curve; $\frac{1}{2}$, bold dashed curve; $\frac{1}{4}$, solid curve.

tion will be $\tilde{y}_c = [-A\gamma/(\gamma t+1) + (D/\gamma^2)(\gamma t+1)^2]e^{-\beta t}$. There is a difficulty in this case since $\tilde{y}_+ = A^- y_+ = [-A\gamma/(\gamma t+1)]e^{-\beta t}$, whereas $\tilde{y}_- = A^- y_- = [B/(\gamma t+1)]e^{-\beta t} \propto \tilde{y}_+$. To find the independent $\tilde{y}_-$ solution we write $\tilde{y}_- = z(t)\tilde{y}_+$ and determine the function $z(t)$ from $\tilde{N}_g \tilde{y}_- = 0$. The result is $z(t) = C(\gamma t+1)^3/\gamma^3$, where C is an arbitrary constant, and therefore $\tilde{y}_- = D[(\gamma t+1)^2/\gamma^2]e^{-\beta t}$, D being another arbitrary constant.

(iii) For overdamping $\beta^2 > \omega_0^2$, the initial free general solution is $y_0 = \tilde{A}_0 e^{-\beta t}\cosh(\alpha t + \phi)$, whereas the γ solution is $\tilde{y}_0 = -\tilde{A}_0 e^{-\beta t}(\alpha \sinh(\alpha t+\phi) - [\gamma/(\gamma t+1)]\cosh(\alpha t+\phi))$.

Plots corresponding to these cases are presented in Figs. 1–3. We note that in the limit $\gamma \to 0$ the modes $\tilde{y}_{\omega,\beta,\gamma}$ are going to the Newtonian damping modes $y_{\omega,\beta}$ for all three classes of free damping motion. Moreover, we placed ourselves herein in the well-behaved regime of motion, i.e., for time and parameter ranges where the modes do not grow with time and their amplitudes are finite. However, from the point of view of the γ parameter the modes $\tilde{y}$ are always singular, i.e., they blow up at some negative time moment for positive γ and at some positive instant for negative γ. Such blow-up solutions are quite well known in nonlinear physics. On the other hand, even the Newtonian modes $y_{\omega,\beta}$ are growing with time in the past or for negative β in the future (divergent and flutter instabilities are textbook knowledge [5]). What we claim here is that when one starts a damping-type measurement after a ''mechanical'' blow-up phenomenon, Riccati parameter modes may be present. As we said, they may also occur before a blow-up phenomenon (for negative γ), an equally important case. In this situation the Riccati parameter distinguishes them from more common instability modes. Thus an extended Riccati-type parametrization of free damping can indeed be useful. The complexification of the Riccati parameter adds one more parameter to the Riccati damping modes. Depending on the sign of the imaginary part, new contributions to either damping or destabilization of the modes occur.

In summary, what we have obtained here are Riccati parameter families of damping modes related to the Newtonian free damping ones by means of Witten's supersymmetric scheme and the general Riccati solution.

This work was partially supported by CONACyT Project No. 4868-E9406.

[1] W. Hauser, *Introduction to the Principles of Mechanics* (Addison-Wesley, Reading, MA, 1965), pp. 106–113; G. R. Fowles, *Analytical Mechanics* (Saunders College Publishing, New York, 1986), pp. 64–68; J. D. Garrison, Am. J. Phys. **42**, 694 (1974); **43**, 463 (1975).
[2] E. Witten, Nucl. Phys. B **185**, 513 (1981).
[3] B. Mielnik, J. Math. Phys. **25**, 3387 (1984).
[4] G. Darboux, C. R. Acad. Sci. **94**, 1456 (1882).
[5] D. J. Inman, *Engineering Vibration* (Prentice-Hall, Englewood Cliffs, NJ, 1996).